

微分方程式

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1 常微分方程式

§ 1.1 常微分方程式

定義 1.1 (常微分方程式)

$$F(x, y, y', y'', \dots, y^{(n)}) = 0$$

常微分方程式 (ordinary differential equation) typing...

例 1.2 (常微分方程式)

$$\begin{aligned}y'' - y &= x \\2yy''' - (y')^2 &= -4x \\3y' - y &= \sin x\end{aligned}$$

typing...

例 1.3 (常微分方程式)

$$y' = x$$

typing...



定義 1.4 (初期条件, 一般解, 特殊解) 一般解 (general solution) typing...

例 1.5 (初期条件, 一般解, 特殊解) $y' = x$ の typing...

§ 1.2 変数分離型方程式

定義 1.6 (変数分離型)

$$y' = f(x)g(y)$$

変数分離型 typing...

□

例 1.7 (変数分離型)

$$y' = xy$$

typing...

□

例 1.8 (変数分離型)

$$y' = -2xy$$

typing...

初期条件 $y(0) = 1$ のもとでの特殊解は typing...

□

§ 1.3 同次型方程式

例 1.9 (同次型)

$$y' = \frac{x+y}{x-y}$$

typing...

□

例 1.10 (同次型)

$$(x^2 - y^2)y' = 2xy$$

typing...

相似変換 typing...

特異解 $y = 0$ をもつ . typing...

□

§ 1.4 完全微分型方程式

例 1.11 (完全微分型)

$$(3x - 4y + 3)y' + 6x + 3y + 5 = 0$$

typing...

□

例 1.12 (完全微分型)

$$(2x - 2y - 1)y' + 2x + 2y + 1 = 0$$

typing...

例 1.13 (完全微分型) 積分因子を求め,

$$(xy - x^2)y' + 1 - xy = 0$$

typing...

例 1.14 (完全微分型) 積分因子を求め,

$$y' \cos y + \sin y = 0$$

typing...

§ 1.5 演習問題 ~ 変数分離型, 同次型, 完全微分型

問 1.15 (変数分離型) 次の変数分離型方程式の一般解を求め解曲線を図示せよ. また, 初期条件 $y(0) = y_0$ のときの特殊解もそれぞれ求めよ.

- (1) $y' = x$ (2) $y' = \sin x$ (3) $y' = y$ (4) $y' = xy$ (5) $y' = y(1 - y)$ (6) $y' = x(1 - y)$ (7) $y' = 1 - y^2$ (8) $y' = \alpha(1 - y)^2$ (9) $y' + \nu y = g$ (10) $y' = x^2 y$ (11) $yy' = x^2 + 3$ (12) $2xy' = y$ (13) $yy' + x = 0$ (14) $y + xy' = 0$ (15) $xy' + 1 = y$ (16) $xy' = y(y - 1)$ (17) $y' = -\mu y$ (18) $y' = \frac{x}{y}$ (19) $xy' + y = 0$ (20) $y' = \mu y \left(1 - \frac{y}{K}\right)$ (21) $y' = (1 + x) \sec y$ (22) $(x + xy)y' = y - xy$ (23) $(x + 1)y' - x(y^2 + 1) = 0$ (24) $y^2 y' + xy^3 = x$ (25) $y' + y \tan x = 0$ (26) $xy' + y = 2xy$ (27) $y = x(x + 1)y'$ (28) $(1 - x)y' = 1 + y$ (29) $(y + 1)y' + x = 1$ (30) $(1 + x)y + x(1 - y)y' = 0$ (31) $y' \cos^2 x + \sin x \cos^2 y = 0$ (32) $(1 + y)xy' + (1 + x)y = 0$ (33) $y' + y \tan x = 0$ (34) $y' = x + y$ ($z = x + y$ とおく) (35) $y' = ax + by$ ($z = ax + by$ とおく) (36) $y' + 1 = x + 2y$ ($z = x + 2y - 1$ とおく) (37) $y' + 1 = e^{x+y}$ ($z = x + y$ とおく) (38) $y' + ay = bx$ (変数変換要) (39) $y' = y + cx^2$ (変数変換要) □

問 1.16 (同次型) 次の同次型方程式の一般解を求めよ. また, 初期値 $y(0) = y_0$ における特殊解を求めよ. さらには, 解曲線を図示せよ.

- (1) $(x - y) = (x + y)y'$ (2) $y^2 + 2xyy' = x^2$ (3) $2xyy' = x^2 + y^2$ (4) $yy' + x = 0$ (5) $xy' = \alpha y$ (6) $(x^2 - y^2)y' = 2xy$ (7) $yy' + x + 2y = 0$ (8) $xy' + y = 2x$ (9) $(x + y)y' + y = x$ (10) $xyy' + x^2 = y^2$ (11) $2xyy' + x^2 = y^2$ (12) $x(x - y)y' + y^2 = 0$ (13) $2xyy' + y^2 = x^2$ (14) $x + yy' = 2y$ (15) $(x + y)y' + y = x$ (16) $xy' = y + \sqrt{x^2 + y^2}$ (17) $x \cot \frac{y}{x} + xy' = y$ (18) $xy' = y + x \sin \frac{y}{x}$ (19) $(x + y + 4)y' + x - y - 2 = 0$ (20) $yy' + y = 2x$ (21) $(x - y)y' = x + y$ □

問 1.17 (完全微分型) 次の方程式の一般解を求めよ.

- (1) $x + 4y + (4x + 3y)y' = 0$ (2) $y \sin x - y' \cos x = 0$ (3) $2xe^y + 1 + (x^2 e^y + 2y)y' = 0$ (4) $4x^3 + 3x^2 y - 3y^3 + (x^3 - 9xy^2 - 4y^3)y' = 0$ (5) $e^{x/y} + (1 - x/y)e^{x/y}y' = 0$ (6) $x^3 + 4x^3 y^3 + (y^2 + 3x^4 y^2)y' = 0$ (7) $(x - y + 1)y' - x + y + 2 = 0$ (8) $(x^2 + 4xy + 3)y' + e^x + 2xy + 2y^2 = 0$ (9) $(x \cos y + 2y^3)y' + \sin y + 3x^2 - 1 = 0$ (10) $y' + x + 2y = 0$ (11) $(2xy^2 + 3y)y' + y^3 = 0$ (12) $xy' \cos y + \sin y = 0$ (13) $y' \sin x \sinh y + \cos x \cosh y = 0$ (14) $(2y - x - 1)y' + 2x - y + 1 = 0$ (15) $(x^2 - y^2)y' + x^2 + 2xy = 0$ (16) $x^2 - 2y + (y^2 - 2x)y' = 0$ (17) $3(x^2 + x^2 y^2) + 2(y + x^3 y)y' = 0$ (18) $\frac{x}{x^2 - y^2} - \frac{yy'}{x^2 - y^2} = 0$ □

問 1.18 (完全微分型) 次の方程式の積分因子を求め一般解を求めよ.

- (1) $\cos y - y' \sin y = 0$ (2) $2y/x + 1 + y' = 0$ (3) $y^2 - 2xy + (4y^2 + 3xy - 2x^2)y' = 0$ (4) $xy^2 + y^3 + (x^3 + 3x^2 y + xy^2)y' = 0$ (5) $y' + x + 2y = 0$ (6) $(2xy^2 + 3y)y' + y^3 = 0$

- (7) $x(3y^2 + x)y' + y(2y^2 + 3x) = 0$ (8) $xy' + 1 = 0$ (9) $xy' \log |x| + y = 0$
 (10) $y - xy' = 0$ (11) $2y - xy' = 0$ (12) $1 + y^2 + xyy' = 0$ (13) $\cot y - xy' = 0$ □

問 1.19 (直交曲線族) 次の曲線族の直交曲線族を求め図示せよ.

- (1) $x^2 + y^2 = r^2$ (2) $\left(\frac{x}{3a}\right)^2 + \left(\frac{y}{2a}\right)^2 = 1$ (3) $\left(\frac{x}{2a}\right)^2 - \left(\frac{y}{a}\right)^2 = 1$
 (4) $x^2 + y^2 - 2rx = 0$ □

注意 1.20 (三角関数の逆数)

$$\operatorname{cosec} x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{1}{\tan x}$$

□

§ 1.6 定数係数齊次線形方程式

例 1.21 (定数係数齊次線形方程式)

$$y'' - 3y' + 2y = 0$$

typing...



例 1.22 (定数係数齊次線形方程式)

$$y'' + 2y' + y = 0$$

typing...



例 1.23 (定数係数齊次線形方程式)

$$y'' - 4y' + 13y = 0$$

typing...



§ 1.7 定数係数非齊次線形方程式

例 1.24 (定数係数非齊次線形方程式)

$$y'' - 3y' + 2y = x$$

typing...



§ 1.8 演習問題 ~ 定数係数線形方程式

問 1.25 (定数係数斉次線形方程式) 次の方程式の一般解を求めよ.

- (1) $y'' - y = 0$ (2) $y'' - y' - 2y = 0$ (3) $y'' - y' - 6y = 0$ (4) $y'' + y = 0$
 (5) $y'' + y' - 2y = 0$ (6) $y'' + y' - 6y = 0$ (7) $3y'' - y' - 2y = 0$ (8) $y'' - 2y' + 2y = 0$
 (9) $y'' - 2y' - 3y = 0$ (10) $y'' - 2y' + 5y = 0$ (11) $2y'' - 2y' + y = 0$
 (12) $2y'' - 2y' + 5y = 0$ (13) $y'' + 2y' - y = 0$ (14) $y'' + 2y' + y = 0$
 (15) $y'' + 2y' + 2y = 0$ (16) $y'' + 2y' - 3y = 0$ (17) $y'' + 2y' + 2y = 0$
 (18) $y'' + 2y' + 5y = 0$ (19) $y'' + 2y' + 10y = 0$ (20) $2y'' + 2y' + y = 0$
 (21) $3y'' + 2y' - y = 0$ (22) $y'' - 3y' + 2y = 0$ (23) $y'' - 3y' + 3y = 0$
 (24) $y'' + 3y' - 4y = 0$ (25) $2y'' + 3y' + y = 0$ (26) $y'' - 4y = 0$ (27) $y'' - 4y' + 3y = 0$
 (28) $y'' - 4y' + 4y = 0$ (29) $y'' + 4y = 0$ (30) $y'' + 4y' + 4y = 0$ (31) $y'' + 4y' - 5y = 0$
 (32) $y'' + 5y' - 6y = 0$ (33) $2y'' - 5y' + 3y = 0$ (34) $2y'' - 5y' + 4y = 0$
 (35) $y'' - 6y' + 8y = 0$ (36) $y'' - 6y' + 9y = 0$ (37) $y'' - 6y' + 13y = 0$
 (38) $y'' + 6y' + 9y = 0$ (39) $2y'' + 6y' + 5y = 0$ (40) $y'' - 7y' + 12y = 0$
 (41) $y'' - 8y' + 15y = 0$ (42) $y'' - 8y' + 16y = 0$ (43) $y'' - 8y' + 32y = 0$
 (44) $y'' + 9y = 0$ (45) $4y'' - 12y' + 9y = 0$ (46) $y'' - ay' = 0$ ($a \neq 0$)
 (47) $y'' + a^2y = 0$ ($a \neq 0$) (48) $y'' - 2\omega y' + \omega^2 y = 0$ (49) $y'' + (\alpha + \beta)y' + \alpha\beta y = 0$
 (50) $y''' - 3y'' + 4y = 0$ (51) $y''' + 3y'' + 4y' + 2y = 0$ (52) $y'' + 2y' - 3y = 0$
 (53) $y''' + 3y'' + 3y' + y = 0$ (54) $y''' - 3y'' + 2y' = 0$ (55) $y''' + y'' - 2y' + 12y = 0$
 (56) $y''' + y'' + y' - 3y = 0$ (57) $y''' + 3y'' + 3y' + y = 0$ (58) $y''' - 3y'' + 2y' = 0$
 (59) $y''' - 6y'' + 11y' - 6y = 0$ (60) $y''' - 2y'' - 5y' + 6y = 0$ (61) $y''' - 3y'' + 3y' - y = 0$

□

問 1.26 (定数係数非斉次線形方程式) 次の方程式の一般解を求めよ.

- (1) $y' - y = \sin x$ (2) $y' + y = x^2 + x - 1$ (3) $y' - 2y = xe^{2x}$ (4) $y'' - y' = e^{2x}$
 (5) $y'' - y' - 2y = x$ (6) $y'' - y' - 2y = x^2 - x$ (7) $y'' - y' - 2y = \sin x$ (8) $y'' + y' = e^{-x}$
 (9) $y'' + y = cx \sin x$ (10) $y'' + y' - 2y = be^x$ (11) $y'' + y' - 2y = be^{-x}$
 (12) $y'' + y' - 6y = e^{3x}$ (13) $y'' + y' - 6y = \cos x$ (14) $y'' - 2y' + y = e^x$
 (15) $y'' - 2y' + 2y = x^2 - 1$ (16) $y'' - 2y' + 2y = e^x \sin x$ (17) $y'' - 2y' + 2y = 2e^x \cos x$
 (18) $y'' - 2y' - 3y = e^{-x}$ (19) $y'' - 2y' - 3y = e^{-x} + x$ (20) $y'' - 2y' + 5y = x$
 (21) $y'' - 2y' + 5y = e^x$ (22) $y'' - 2y' + 5y = \sin x$ (23) $y'' + 2y' - 3y = x^2$
 (24) $y'' + 2y' - 3y = e^{2x}$ (25) $y'' + 2y' - 3y = \cos x$ (26) $y'' + 2y' - 3y = e^x \sin x$
 (27) $y'' + 2y' - 8y = e^{2x}$ (28) $y'' - 3y' + 2y = 2$ (29) $y'' - 3y' + 2y = 2x - 1$
 (30) $y'' - 3y' + 2y = e^x$ (31) $y'' - 3y' + 2y = \cos x$ (32) $y'' - 3y' + 2y = \cos 2x$
 (33) $y'' - 3y' + 2y = 2 \cos x + 3e^x$ (34) $y'' - 3y' + 2y = xe^x$ (35) $y'' - 3y' + 3y = e^x$
 (36) $y'' + 3y' = e^{3x} + x$ (37) $2y'' + 3y' + y = ax$ (38) $y'' - 4y' = x + x^2 + \sin x$
 (39) $y'' - 4y' + 3y = x$ (40) $y'' - 4y' + 3y = x^2$ (41) $y'' - 4y' + 3y = \cos x$
 (42) $y'' - 4y' + 3y = 3e^{2x} + 4e^x + 2e^{3x}$ (43) $y'' - 4y' + 4y = e^x$
 (44) $y'' + 4y' + y = e^x$ (45) $y'' - 5y' - 4y = x + x^2 + \sin x$ (46) $y'' - 6y' + 8y = e^{3x}$

- (47) $y'' - 6y' + 9y = be^{3x}$ (48) $y'' + 8y' + 17y = 2e^{-3x}$ (49) $y'' - y = x$
(50) $y'' - y = e^x$ (51) $y'' - y = e^{2x}$ (52) $y'' - y = \sin x$ (53) $y'' - y = xe^x$
(54) $y'' - y = e^x \sin x$ (55) $y'' + y = x$ (56) $y'' + y = \sin x$ (57) $y'' + y = \cos x$
(58) $y'' + y = 2 \cos x$ (59) $y'' + y = x + x^2 + \sin x$ (60) $y'' + y = x \sin x$
(61) $y'' + y = e^x \sin x$ (62) $y'' + 2y = x^2$ (63) $y'' - 4y = 1$ (64) $y'' - 4y = x$
(65) $y'' - 4y = \sin x$ (66) $y'' - 4y = \cos x$ (67) $y'' - 4y = e^{2x}$ (68) $y'' - 4y = e^{-2x}$
(69) $y'' - \omega^2 y = a + bx$ (70) $y'' - \omega^2 y = ce^{\rho x}$ ($\rho \neq \pm \omega$) (71) $y'' - \omega^2 y = ce^{\omega x}$
(72) $y'' - \omega^2 y = a + bx + ce^{\rho x}$ (73) $y'' + \omega^2 y = bx$
(74) $y'' + \omega^2 y = a \sin \Omega x$ ($\omega \neq \Omega$) (75) $y''' - y = e^{2x}$ (76) $y''' - 7y' + 6y = e^x$
(77) $y''' + y'' + y' + y = e^x$ (78) $y''' - 3y'' + 4y = e^{2x}$ (79) $y''' - 5y'' + 6y' = e^{2x}$
(80) $y''' - 5y'' + 2y' + 8y = 16x$ □

問 1.27 (オイラーの方程式) 次の方程式を $x = e^t$ と変数変換して一般解を求めよ .

- (1) $x^2 y'' - 2y = 0$ (2) $x^2 y'' - xy' + y = 0$ (3) $x^2 y'' - xy' + y = 0$
(4) $x^2 y'' - xy' - 3y = 0$ (5) $x^2 y'' - xy' + 5y = 0$ (6) $x^2 y'' - xy' + y = (\log x)^2$
(7) $xy'' - 2y' = 0$ (8) $x^2 y'' - 2xy' + 2y = x^3$ (9) $x^2 y'' - 3xy' + 3y = 0$
(10) $x^2 y'' - 3xy' + 3y = x^3 + x^4$ (11) $x^2 y'' - 3xy' + 3y = x^5 \sin x$ (12) $x^2 y'' - 3xy' + 4y = 0$
(13) $x^2 y'' - 3xy' + 4y = x^3 + x^4$ (14) $x^2 y'' - 3xy' + 4y = x^2 + x^2 \log x$
(15) $x^2 y'' - 4xy' + 6y = 2x$ (16) $x^2 y'' - 4xy' + 6y = 0$ (17) $x^2 y y'' + x^2 y'^2 - x y y' = 0$
(18) $x^2 y'' - (2m - 1)xy' + m^2 y = 0$ (19) $x^2 y'' - (\alpha + \beta - 1)xy' + \alpha \beta y = 0$
(20) $y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$ (21) $y'' - \frac{\alpha}{x} y' + \frac{\alpha}{x^2} y = 0$
(22) $y'' - \frac{y'}{x} + \frac{y}{x^2} = bx^\alpha$ ($\mathbb{R} \ni \alpha \neq 1$) □

§ 1.9 1 階線形方程式

例 1.28 (1 階線形方程式)

$$y' - y = x$$

typing...

□

例 1.29 (1 階線形方程式)

$$y' - ay = q(x)$$

typing...

□

§ 1.10 ベルヌーイの方程式

例 1.30 (ベルヌーイの方程式)

$$y' + y = xy^3$$

typing...

□

§ 1.11 演習問題 ~ 1 階線形, ベルヌーイ型

問 1.31 (同次型) 次の方程式の一般解を求めよ .

- (1) $y' = y + e^{2x}$ (2) $xy' = 2y + x(x+2)$ (3) $y' + 2xy = x$ (4) $xy' = x + y$
(5) $y' = y + xy^2$ (6) $y' = 2y + e^{2x}y^3$ (7) $y' + y = x$ (8) $xy' + y = x$
(9) $xy' + y = x^2$ (10) $y' = y + \sin x$ (11) $y' = xy + x$ (12) $y' \cos x + y \sin x = 1$
(13) $xy' + y = \cos x$ (14) $y' + \cos^2 y \tan y = x \cos^2 y$

□

§ 1.12 演習問題 ~ 巾級数展開

問 1.32 (巾級数展開) 次の方程式の一般解を $x = 0$ に巾級数展開で求めよ.

- (1) $y'' - 3y' + 2y = 0$ (2) $y'' - xy' - y = 0$ (3) $(x^2 - 1)y'' - 2y = 0$
(4) $(1 - x^2)y'' - 2xy' + 2y = 0$ (5) $y' = 1 + x + 2y$ (6) $xy' = x^2 + y$
(7) $y'' - 2xy' - 2y = 0$ (8) $(x^2 - 1)y'' + 2xy' - 2y = 0$ (9) $(1 + x^2)y'' - xy' - 8y = 0$
(10) $y'' + x^2y' + xy = 0$ (11) $y' - 2y = 1 + x$ (12) $y' - y/x = x$ (13) $y'' - xy = 0$
(14) $(1 - x^2)y' - xy' + y = 0$ (15) $(1 - x)y' = -1 + y$ (16) $(1 + x^2)y'' + xy' - y = 0$
(17) $y' - y + x = 0$ (18) $y' - 2xy = x$ (19) $y' + y = x^2$ (20) $(x + 1)y' + y = 2x + 3x^2$
(21) $y'' + y = 0$ (22) $y'' - y = 2e^x$ (23) $(1 - x)y'' + xy' - y = 0$
(24) $(1 + x^2)y'' - 2xy' + 2y = 0$ □

問 1.33 (確定特異点をもつ方程式の巾級数展開) 次の方程式の一般解を $x = 0$ に巾級数展開で求めよ.

- (1) $4xy'' + 2y' + y = 0$ (2) $x^2y'' + (x^2 - 3x)y' + (4 - 2x)y = 0$
(3) $x^2(1 - x)y'' - x(2 + x)y' + 2y = 0$ (4) $xy'' + y' + xy = 0$ (5) $2xy'' + (1 - 2x)y' - y = 0$
(6) $xy'' + y' - y = 0$ (7) $x^2y'' + xy' + (x^2 - 1/4)y = 0$ (8) $xy'' - xy' - y = 0$ □