微分方程式

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1 常微分方程式

§ 1.1 常微分方程式

定義 1.1 **（常微分方程式）**

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0$$

常微分方程式（ordinary differential equation）typing... □

例 1.2 **（常微分方程式）**

$$y'' - y = x$$
$$2yy''' - (y')^2 = -4x$$
$$3y' - y = \sin x$$

typing... □

例 1.3 **（常微分方程式）**

$$y' = x$$

typing... □

Figure. typing...

定義 1.4 **（初期条件，一般解，特殊解）** **一般解（general solution）**typing... □

例 1.5 **（初期条件，一般解，特殊解）** $y' = x$ の typing... □

§ 1.2 変数分離型方程式

定義 1.6 **（変数分離型）**

$$y' = f(x)g(y)$$

変数分離型 typing... □

例 1.7 **（変数分離型）**

$$y' = xy$$

typing... □

例 1.8 **（変数分離型）**

$$y' = -2xy$$

typing...
初期条件 $y(0) = 1$ のもとでの特殊解は typing... □

§ 1.3 同次型方程式

例 1.9 **（同次型）**

$$y' = \frac{x + y}{x - y}$$

typing... □

例 1.10 **（同次型）**

$$(x^2 - y^2)y' = 2xy$$

typing...
相似変換 typing...
特異解 $y = 0$ をもつ．typing... □

§ 1.4 完全微分型方程式

例 1.11 **（完全微分型）**

$$(3x - 4y + 3)y' + 6x + 3y + 5 = 0$$

typing... □

例 1.12 **(完全微分型)**

$$(2x-2y-1)y'+2x+2y+1=0$$

typing... □

例 1.13 **(完全微分型)** 積分因子を求め,

$$(xy-x^2)y'+1-xy=0$$

typing... □

例 1.14 **(完全微分型)** 積分因子を求め,

$$y'\cos y+\sin y=0$$

typing... □

§ 1.5 演習問題 〜 変数分離型，同次型，完全微分型

問 1.15 **（変数分離型）** 次の変数分離型方程式の一般解を求め解曲線を図示せよ．また，初期条件 $y(0) = y_0$ のときの特殊解もそれぞれ求めよ．

(1) $y' = -\mu y$ (2) $y' = \dfrac{x}{y}$ (3) $xy' + y = 0$ (4) $y' = \mu y\left(1 - \dfrac{y}{K}\right)$
(5) $y' = (1+x)\sec y$ (6) $(x + xy)y' = y - xy$ (7) $(x+1)y' - x(y^2+1) = 0$
(8) $y^2y' + xy^3 = x$ (9) $y' + y\tan x = 0$ (10) $y' = ax + by$ ($z = ax + by$ とおく)
(11) $y' = x^2y$ (12) $xy' + y = 2xy$ (13) $y = x(x+1)y'$ (14) $(1-x)y' = 1 + y$
(15) $y' + 1 = x + 2y$ ($z = x + 2y - 1$ とおく) (16) $y' + 1 = e^{x+y}$ ($z = x + y$ とおく)
(17) $2xy' = y$ (18) $yy' + x = 0$ (19) $y + xy' = 0$ (20) $xy' + 1 = y$
(21) $(y+1)y' + x = 1$ (22) $yy' = x^2 + 3$ (23) $(1+x)y + x(1-y)y' = 0$
(24) $y'\cos^2 x + \sin x\cos^2 y = 0$ (25) $(1+y)xy' + (1+x)y = 0$ (26) $y' + y\tan x = 0$
(27) $xy' = y(y-1)$ (28) $y' + ay = bx$ (変数変換要) (29) $y' = y + cx^2$ (変数変換要) □

問 1.16 **（同次型）** 次の同次型方程式の一般解を求めよ．また，初期値 $y(0) = y_0$ における特殊解を求めよ．さらには，解曲線を図示せよ．

(1) $yy' + x = 0$ (2) $xy' = \alpha y$ (3) $(x^2 - y^2)y' = 2xy$ (4) $yy' + x + 2y = 0$
(5) $xy' + y = 2x$ (6) $(x+y)y' + y = x$ (7) $xyy' + x^2 = y^2$ (8) $2xyy' + x^2 = y^2$
(9) $x(x-y)y' + y^2 = 0$ (10) $2xyy' + y^2 = x^2$ (11) $x + yy' = 2y$ (12) $(x+y)y' + y = x$
(13) $xy' = y + \sqrt{x^2+y^2}$ (14) $x\cot\dfrac{y}{x} + xy' = y$ (15) $xy' = y + x\sin\dfrac{y}{x}$
(16) $(x+y+4)y' + x - y - 2 = 0$ (17) $yy' + y = 2x$ (18) $(x-y)y' = x + y$ □

問 1.17 **（完全微分型）** 次の方程式の一般解を求めよ．

(1) $x + 4y + (4x + 3y)y' = 0$ (2) $y\sin x - y'\cos x = 0$ (3) $2xe^y + 1 + (x^2e^y + 2y)y' = 0$
(4) $4x^3 + 3x^2y - 3y^3 + (x^3 - 9xy^2 - 4y^3)y' = 0$ (5) $e^{x/y} + (1 - x/y)e^{x/y}y' = 0$
(6) $x^3 + 4x^3y^3 + (y^2 + 3x^4y^2)y' = 0$ (7) $(x - y + 1)y' - x + y + 2 = 0$
(8) $(x^2 + 4xy + 3)y' + e^x + 2xy + 2y^2 = 0$ (9) $(x\cos y + 2y^3)y' + \sin y + 3x^2 - 1 = 0$
(10) $y' + x + 2y = 0$ (11) $(2xy^2 + 3y)y' + y^3 = 0$ (12) $xy'\cos y + \sin y = 0$
(13) $y'\sin x\sinh y + \cos x\cosh y = 0$ (14) $(2y - x - 1)y' + 2x - y + 1 = 0$
(15) $(x^2 - y^2)y' + x^2 + 2xy = 0$ (16) $x^2 - 2y + (y^2 - 2x)y' = 0$
(17) $3(x^2 + x^2y^2) + 2(y + x^3y)y' = 0$ (18) $\dfrac{x}{x^2 - y^2} - \dfrac{yy'}{x^2 - y^2} = 0$ □

問 1.18 **（完全微分型）** 次の方程式の積分因子を求め一般解を求めよ．

(1) $\cos y - y'\sin y = 0$ (2) $2y/x + 1 + y' = 0$ (3) $y^2 - 2xy + (4y^2 + 3xy - 2x^2)y' = 0$
(4) $xy^2 + y^3 + (x^3 + 3x^2y + xy^2)y' = 0$ (5) $y' + x + 2y = 0$ (6) $(2xy^2 + 3y)y' + y^3 = 0$
(7) $x(3y^2 + x)y' + y(2y^2 + 3x) = 0$ (8) $xy' + 1 = 0$ (9) $xy'\log|x| + y = 0$
(10) $y - xy' = 0$ (11) $2y - xy' = 0$ (12) $1 + y^2 + xyy' = 0$ (13) $\cot y - xy' = 0$ □

問 1.19 **（直交曲線族）** 次の曲線族の直交曲線族を求め図示せよ．

(1) $x^2 + y^2 = r^2$ (2) $\left(\dfrac{x}{3a}\right)^2 + \left(\dfrac{y}{2a}\right)^2 = 1$ (3) $\left(\dfrac{x}{2a}\right)^2 - \left(\dfrac{y}{a}\right)^2 = 1$

(4) $x^2 + y^2 - 2rx = 0$

□

注意 1.20 **（三角関数の逆数）**

$$\operatorname{cosec} x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{1}{\tan x}$$

□

§ 1.6 定数係数斉次線形方程式

例 1.21 **（定数係数斉次線形方程式）**

$$y'' - 3y' + 2y = 0$$

typing... □

例 1.22 **（定数係数斉次線形方程式）**

$$y'' + 2y' + y = 0$$

typing... □

例 1.23 **（定数係数斉次線形方程式）**

$$y'' - 4y' + 13y = 0$$

typing... □

§ 1.7 定数係数非斉次線形方程式

例 1.24 **（定数係数非斉次線形方程式）**

$$y'' - 3y' + 2y = x$$

typing... □

§ 1.8 演習問題 〜 定数係数線形方程式

問 1.25 **(定数係数斉次線形方程式)** 次の方程式の一般解を求めよ.

(1) $y'' - y = 0$ (2) $y'' - y' - 2y = 0$ (3) $y'' - y' - 6y = 0$ (4) $y'' + y = 0$
(5) $y'' + y' - 2y = 0$ (6) $y'' + y' - 6y = 0$ (7) $3y'' - y' - 2y = 0$ (8) $y'' - 2y' + 2y = 0$
(9) $y'' - 2y' - 3y = 0$ (10) $y'' - 2y' + 5y = 0$ (11) $2y'' - 2y' + y = 0$
(12) $2y'' - 2y' + 5y = 0$ (13) $y'' + 2y' - y = 0$ (14) $y'' + 2y' + y = 0$
(15) $y'' + 2y' + 2y = 0$ (16) $y'' + 2y' - 3y = 0$ (17) $y'' + 2y' + 2y = 0$
(18) $y'' + 2y' + 5y = 0$ (19) $y'' + 2y' + 10y = 0$ (20) $2y'' + 2y' + y = 0$
(21) $3y'' + 2y' - y = 0$ (22) $y'' - 3y' + 2y = 0$ (23) $y'' - 3y' + 3y = 0$
(24) $y'' + 3y' - 4y = 0$ (25) $2y'' + 3y' + y = 0$ (26) $y'' - 4y = 0$ (27) $y'' - 4y' + 3y = 0$
(28) $y'' - 4y' + 4y = 0$ (29) $y'' + 4y = 0$ (30) $y'' + 4y' + 4y = 0$ (31) $y'' + 4y' - 5y = 0$
(32) $y'' + 5y' - 6y = 0$ (33) $2y'' - 5y' + 3y = 0$ (34) $2y'' - 5y' + 4y = 0$
(35) $y'' - 6y' + 8y = 0$ (36) $y'' - 6y' + 9y = 0$ (37) $y'' - 6y' + 13y = 0$
(38) $y'' + 6y' + 9y = 0$ (39) $2y'' + 6y' + 5y = 0$ (40) $y'' - 7y' + 12y = 0$
(41) $y'' - 8y' + 15y = 0$ (42) $y'' - 8y' + 16y = 0$ (43) $y'' - 8y' + 32y = 0$
(44) $y'' + 9y = 0$ (45) $4y'' - 12y' + 9y = 0$ (46) $y'' - ay' = 0\ (a \neq 0)$
(47) $y'' + a^2 y = 0\ (a \neq 0)$ (48) $y'' - 2\omega y' + \omega^2 y = 0$ (49) $y'' + (\alpha + \beta)y' + \alpha\beta\, y = 0$
(50) $y''' - 3y'' + 4y = 0$ (51) $y''' + 3y'' + 4y' + 2y = 0$ (52) $y'' + 2y' - 3y = 0$
(53) $y''' + 3y'' + 3y' + y = 0$ (54) $y''' - 3y'' + 2y' = 0$ (55) $y''' + y'' - 2y' + 12y = 0$
(56) $y''' + y'' + y' - 3y = 0$ (57) $y''' + 3y'' + 3y' + y = 0$ (58) $y''' - 3y'' + 2y' = 0$
(59) $y''' - 6y'' + 11y' - 6y = 0$ (60) $y''' - 2y'' - 5y' + 6y = 0$ (61) $y''' - 3y'' + 3y' - y = 0$

□

問 1.26 **(定数係数非斉次線形方程式)** 次の方程式の一般解を求めよ.

(1) $y' - y = \sin x$ (2) $y' + y = x^2 + x - 1$ (3) $y' - 2y = xe^{2x}$ (4) $y'' - y' = e^{2x}$
(5) $y'' - y' - 2y = x$ (6) $y'' - y' - 2y = x^2 - x$ (7) $y'' - y' - 2y = \sin x$ (8) $y'' + y' = e^{-x}$
(9) $y'' + y = cx \sin x$ (10) $y'' + y' - 2y = be^x$ (11) $y'' + y' - 2y = be^{-x}$
(12) $y'' + y' - 6y = e^{3x}$ (13) $y'' + y' - 6y = \cos x$ (14) $y'' - 2y' + y = e^x$
(15) $y'' - 2y' + 2y = x^2 - 1$ (16) $y'' - 2y' + 2y = e^x \sin x$ (17) $y'' - 2y' + 2y = 2e^x \cos x$
(18) $y'' - 2y' - 3y = e^{-x}$ (19) $y'' - 2y' - 3y = e^{-x} + x$ (20) $y'' - 2y' + 5y = x$
(21) $y'' - 2y' + 5y = e^x$ (22) $y'' - 2y' + 5y = \sin x$ (23) $y'' + 2y' - 3y = x^2$
(24) $y'' + 2y' - 3y = e^{2x}$ (25) $y'' + 2y' - 3y = \cos x$ (26) $y'' + 2y' - 3y = e^x \sin x$
(27) $y'' + 2y' - 8y = e^{2x}$ (28) $y'' - 3y' + 2y = 2$ (29) $y'' - 3y' + 2y = 2x - 1$
(30) $y'' - 3y' + 2y = e^x$ (31) $y'' - 3y' + 2y = \cos x$ (32) $y'' - 3y' + 2y = \cos 2x$
(33) $y'' - 3y' + 2y = 2\cos x + 3e^x$ (34) $y'' - 3y' + 2y = xe^x$ (35) $y'' - 3y' + 3y = e^x$
(36) $y'' + 3y' = e^{3x} + x$ (37) $2y'' + 3y' + y = ax$ (38) $y'' - 4y' = x + x^2 + \sin x$
(39) $y'' - 4y' + 3y = x$ (40) $y'' - 4y' + 3y = x^2$ (41) $y'' - 4y' + 3y = \cos x$
(42) $y'' - 4y' + 3y = 3e^{2x} + 4e^x + 2e^{3x}$ (43) $y'' - 4y' + 4y = e^x$
(44) $y'' + 4y' + y = e^x$ (45) $y'' - 5y' - 4y = x + x^2 + \sin x$ (46) $y'' - 6y' + 8y = e^{3x}$

(47) $y'' - 6y' + 9y = be^{3x}$ (48) $y'' + 8y' + 17y = 2e^{-3x}$ (49) $y'' - y = x$
(50) $y'' - y = e^x$ (51) $y'' - y = e^{2x}$ (52) $y'' - y = \sin x$ (53) $y'' - y = xe^x$
(54) $y'' - y = e^x \sin x$ (55) $y'' + y = x$ (56) $y'' + y = \sin x$ (57) $y'' + y = \cos x$
(58) $y'' + y = 2\cos x$ (59) $y'' + y = x + x^2 + \sin x$ (60) $y'' + y = x\sin x$
(61) $y'' + y = e^x \sin x$ (62) $y'' + 2y = x^2$ (63) $y'' - 4y = 1$ (64) $y'' - 4y = x$
(65) $y'' - 4y = \sin x$ (66) $y'' - 4y = \cos x$ (67) $y'' - 4y = e^{2x}$ (68) $y'' - 4y = e^{-2x}$
(69) $y'' - \omega^2 y = a + bx$ (70) $y'' - \omega^2 y = ce^{\rho x}\ (\rho \neq \pm\omega)$ (71) $y'' - \omega^2 y = ce^{\omega x}$
(72) $y'' - \omega^2 y = a + bx + ce^{\rho x}$ (73) $y'' + \omega^2 y = bx$
(74) $y'' + \omega^2 y = a\sin\Omega x\ (\omega \neq \Omega)$ (75) $y''' - y = e^{2x}$ (76) $y''' - 7y' + 6y = e^x$
(77) $y''' + y'' + y' + y = e^x$ (78) $y''' - 3y'' + 4y = e^{2x}$ (79) $y''' - 5y'' + 6y' = e^{2x}$
(80) $y''' - 5y'' + 2y' + 8y = 16x$ □

問 1.27 **（オイラーの方程式）** 次の方程式を $x = e^t$ と変数変換して一般解を求めよ.
(1) $x^2y'' - 2y = 0$ (2) $x^2y'' - xy' + y = 0$ (3) $x^2y'' - xy' + y = 0$
(4) $x^2y'' - xy' - 3y = 0$ (5) $x^2y'' - xy' + 5y = 0$ (6) $x^2y'' - xy' + y = (\log x)^2$
(7) $xy'' - 2y' = 0$ (8) $x^2y'' - 2xy' + 2y = x^3$ (9) $x^2y'' - 3xy' + 3y = 0$
(10) $x^2y'' - 3xy' + 3y = x^3 + x^4$ (11) $x^2y'' - 3xy' + 3y = x^5 \sin x$ (12) $x^2y'' - 3xy' + 4y = 0$
(13) $x^2y'' - 3xy' + 4y = x^3 + x^4$ (14) $x^2y'' - 3xy' + 4y = x^2 + x^2\log x$
(15) $x^2y'' - 4xy' + 6y = 2x$ (16) $x^2y'' - 4xy' + 6y = 0$ (17) $x^2yy'' + x^2y'^2 - xyy' = 0$
(18) $x^2y'' - (2m-1)xy' + m^2y = 0$ (19) $x^2y'' - (\alpha+\beta-1)xy' + \alpha\beta y = 0$
(20) $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$ (21) $y'' - \frac{\alpha}{x}y' + \frac{\alpha}{x^2}y = 0$
(22) $y'' - \frac{a}{x}y' + \frac{a}{x^2}y = bx^\alpha\ (\mathbb{R} \ni \alpha \neq 1)$ □

§ 1.9 1 階線形方程式

例 1.28 **(1 階線形方程式)**

$$y' - y = x$$

typing... □

例 1.29 **(1 階線形方程式)**

$$y' - ay = q(x)$$

typing... □

§ 1.10 ベルヌーイの方程式

例 1.30 **(ベルヌーイの方程式)**

$$y' + y = xy^3$$

typing... □

§ 1.11 演習問題 〜 1 階線形，ベルヌーイ型

問 1.31 **(同次型)** 次の方程式の一般解を求めよ.

(1) $y' = y + e^{2x}$ (2) $xy' = 2y + x(x+2)$ (3) $y' + 2xy = x$ (4) $xy' = x + y$
(5) $y' = y + xy^2$ (6) $y' = 2y + e^{2x}y^3$ (7) $y' + y = x$ (8) $xy' + y = x$
(9) $xy' + y = x^2$ (10) $y' = y + \sin x$ (11) $y' = xy + x$ (12) $y' \cos x + y \sin x = 1$
(13) $xy' + y = \cos x$ (14) $y' + \cos^2 y \tan y = x \cos^2 y$ □

§ 1.12 演習問題 〜 巾級数展開

問 1.32 **（巾級数展開）** 次の方程式の一般解を $x=0$ に巾級数展開で求めよ.

(1) $y''-3y'+2y=0$ (2) $y''-xy'-y=0$ (3) $(x^2-1)y''-2y=0$
(4) $(1-x^2)y''-2xy'+2y=0$ (5) $y'=1+x+2y$ (6) $xy'=x^2+y$
(7) $y''-2xy'-2y=0$ (8) $(x^2-1)y''+2xy'-2y=0$ (9) $(1+x^2)y''-xy'-8y=0$
(10) $y''+x^2y'+xy=0$ (11) $y'-2y=1+x$ (12) $y'-y/x=x$ (13) $y''-xy=0$
(14) $(1-x^2)y'-xy'+y=0$ (15) $(1-x)y'=-1+y$ (16) $(1+x^2)y''+xy'-y=0$
(17) $y'-y+x=0$ (18) $y'-2xy=x$ (19) $y'+y=x^2$ (20) $(x+1)y'+y=2x+3x^2$
(21) $y''+y=0$ (22) $y''-y=2e^x$ (23) $(1-x)y''+xy'-y=0$
(24) $(1+x^2)y''-2xy'+2y=0$ □

問 1.33 **（確定特異点をもつ方程式の巾級数展開）** 次の方程式の一般解を $x=0$ に巾級数展開で求めよ.

(1) $4xy''+2y'+y=0$ (2) $x^2y''+(x^2-3x)y'+(4-2x)y=0$
(3) $x^2(1-x)y''-x(2+x)y'+2y=0$ (4) $xy''+y'+xy=0$ (5) $2xy''+(1-2x)y'-y=0$
(6) $xy''+y'-y=0$ (7) $x^2y''+xy'+(x^2-1/4)y=0$ (8) $xy''-xy'-y=0$ □