

# 微分方程式

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# 1 常微分方程式

## § 1.1 常微分方程式

**定義 1.1** (常微分方程式)

$$F(x, y, y', y'', \dots, y^{(n)}) = 0$$

常微分方程式 (ordinary differential equation) typing...

**例 1.2** (常微分方程式)

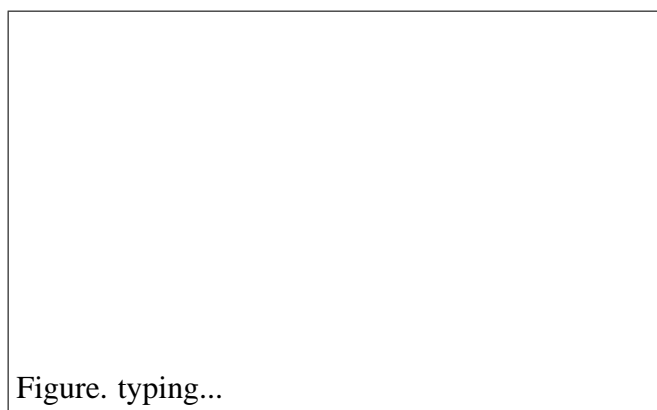
$$\begin{aligned}y'' - y &= x \\2yy''' - (y')^2 &= -4x \\3y' - y &= \sin x\end{aligned}$$

typing...

**例 1.3** (常微分方程式)

$$y' = x$$

typing...



**定義 1.4** (初期条件, 一般解, 特殊解) 一般解 (general solution) typing...

**例 1.5** (初期条件, 一般解, 特殊解)  $y' = x$  の typing...

## § 1.2 変数分離型方程式

**定義 1.6** (変数分離型)

$$y' = f(x)g(y)$$

変数分離型 typing...

□

**例 1.7** (変数分離型)

$$y' = xy$$

typing...

□

**例 1.8** (変数分離型)

$$y' = -2xy$$

typing...

初期条件  $y(0) = 1$  のもとでの特殊解は typing...

□

## § 1.3 同次型方程式

**例 1.9** (同次型)

$$y' = \frac{x+y}{x-y}$$

typing...

□

**例 1.10** (同次型)

$$(x^2 - y^2)y' = 2xy$$

typing...

相似変換 typing...

特異解  $y = 0$  をもつ . typing...

□

## § 1.4 完全微分型方程式

**例 1.11** (完全微分型)

$$(3x - 4y + 3)y' + 6x + 3y + 5 = 0$$

typing...

**例 1.12** (完全微分型)

$$(2x - 2y - 1)y' + 2x + 2y + 1 = 0$$

typing...

**例 1.13** (完全微分型) 積分因子を求め ,

$$(xy - x^2)y' + 1 - xy = 0$$

typing...

**例 1.14** (完全微分型) 積分因子を求め ,

$$y' \cos y + \sin y = 0$$

typing...

## § 1.5 演習問題 ~ 変数分離型, 同次型, 完全微分型

**問 1.15** (変数分離型) 次の変数分離型方程式の一般解を求め解曲線を図示せよ. また, 初期条件  $y(0) = y_0$  のときの特殊解もそれぞれ求めよ.

- (1)  $y' = -\mu y$  (2)  $y' = \frac{x}{y}$  (3)  $xy' + y = 0$  (4)  $y' = \mu y \left(1 - \frac{y}{K}\right)$   
 (5)  $y' = (1+x)\sec y$  (6)  $(x+xy)y' = y - xy$  (7)  $(x+1)y' - x(y^2+1) = 0$   
 (8)  $y^2y' + xy^3 = x$  (9)  $y' + y \tan x = 0$  (10)  $y' = ax + by$  ( $z = ax + by$  とおく)  
 (11)  $y' = x^2y$  (12)  $xy' + y = 2xy$  (13)  $y = x(x+1)y'$  (14)  $(1-x)y' = 1+y$   
 (15)  $y' + 1 = x + 2y$  ( $z = x + 2y - 1$  とおく) (16)  $y' + 1 = e^{x+y}$  ( $z = x + y$  とおく)  
 (17)  $2xy' = y$  (18)  $yy' + x = 0$  (19)  $y + xy' = 0$  (20)  $xy' + 1 = y$   
 (21)  $(y+1)y' + x = 1$  (22)  $yy' = x^2 + 3$  (23)  $(1+x)y + x(1-y)y' = 0$   
 (24)  $y' \cos^2 x + \sin x \cos^2 y = 0$  (25)  $(1+y)xy' + (1+x)y = 0$  (26)  $y' + y \tan x = 0$   
 (27)  $xy' = y(y-1)$  (28)  $y' + ay = bx$  (変数変換要) (29)  $y' = y + cx^2$  (変数変換要) □

**問 1.16** (同次型) 次の同次型方程式の一般解を求めよ. また, 初期値  $y(0) = y_0$  における特殊解を求めよ. さらに, 解曲線を図示せよ.

- (1)  $yy' + x = 0$  (2)  $xy' = \alpha y$  (3)  $(x^2 - y^2)y' = 2xy$  (4)  $yy' + x + 2y = 0$   
 (5)  $xy' + y = 2x$  (6)  $(x+y)y' + y = x$  (7)  $xyy' + x^2 = y^2$  (8)  $2xyy' + x^2 = y^2$   
 (9)  $x(x-y)y' + y^2 = 0$  (10)  $2xyy' + y^2 = x^2$  (11)  $x + yy' = 2y$  (12)  $(x+y)y' + y = x$   
 (13)  $xy' = y + \sqrt{x^2 + y^2}$  (14)  $x \cot \frac{y}{x} + xy' = y$  (15)  $xy' = y + x \sin \frac{y}{x}$   
 (16)  $(x+y+4)y' + x - y - 2 = 0$  (17)  $yy' + y = 2x$  (18)  $(x-y)y' = x + y$  □

**問 1.17** (完全微分型) 次の方程式の一般解を求めよ.

- (1)  $x + 4y + (4x + 3y)y' = 0$  (2)  $y \sin x - y' \cos x = 0$  (3)  $2xe^y + 1 + (x^2e^y + 2y)y' = 0$   
 (4)  $4x^3 + 3x^2y - 3y^3 + (x^3 - 9xy^2 - 4y^3)y' = 0$  (5)  $e^{x/y} + (1 - x/y)e^{x/y}y' = 0$   
 (6)  $x^3 + 4x^3y^3 + (y^2 + 3x^4y^2)y' = 0$  (7)  $(x - y + 1)y' - x + y + 2 = 0$   
 (8)  $(x^2 + 4xy + 3)y' + e^x + 2xy + 2y^2 = 0$  (9)  $(x \cos y + 2y^3)y' + \sin y + 3x^2 - 1 = 0$   
 (10)  $y' + x + 2y = 0$  (11)  $(2xy^2 + 3y)y' + y^3 = 0$  (12)  $xy' \cos y + \sin y = 0$   
 (13)  $y' \sin x \sinh y + \cos x \cosh y = 0$  (14)  $(2y - x - 1)y' + 2x - y + 1 = 0$   
 (15)  $(x^2 - y^2)y' + x^2 + 2xy = 0$  (16)  $x^2 - 2y + (y^2 - 2x)y' = 0$   
 (17)  $3(x^2 + x^2y^2) + 2(y + x^3y)y' = 0$  (18)  $\frac{x}{x^2 - y^2} - \frac{yy'}{x^2 - y^2} = 0$  □

**問 1.18** (完全微分型) 次の方程式の積分因子を求め一般解を求めよ.

- (1)  $\cos y - y' \sin y = 0$  (2)  $2y/x + 1 + y' = 0$  (3)  $y^2 - 2xy + (4y^2 + 3xy - 2x^2)y' = 0$   
 (4)  $xy^2 + y^3 + (x^3 + 3x^2y + xy^2)y' = 0$  (5)  $y' + x + 2y = 0$  (6)  $(2xy^2 + 3y)y' + y^3 = 0$   
 (7)  $x(3y^2 + x)y' + y(2y^2 + 3x) = 0$  (8)  $xy' + 1 = 0$  (9)  $xy' \log|x| + y = 0$   
 (10)  $y - xy' = 0$  (11)  $2y - xy' = 0$  (12)  $1 + y^2 + xy' = 0$  (13)  $\cot y - xy' = 0$  □

**問 1.19** (直交曲線族) 次の曲線族の直交曲線族を求め図示せよ.

(1)  $x^2 + y^2 = r^2$  (2)  $\left(\frac{x}{3a}\right)^2 + \left(\frac{y}{2a}\right)^2 = 1$  (3)  $\left(\frac{x}{2a}\right)^2 - \left(\frac{y}{a}\right)^2 = 1$   
(4)  $x^2 + y^2 - 2rx = 0$

□

**注意 1.20** (三角関数の逆数)

$$\operatorname{cosec} x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{1}{\tan x}$$

□

## § 1.6 定数係数齊次線形方程式

**例 1.21** (定数係数齊次線形方程式)

$$y'' - 3y' + 2y = 0$$

typing...



**例 1.22** (定数係数齊次線形方程式)

$$y'' + 2y' + y = 0$$

typing...



**例 1.23** (定数係数齊次線形方程式)

$$y'' - 4y' + 13y = 0$$

typing...





## § 1.7 定数係数非斉次線形方程式

**例 1.24** (定数係数非斉次線形方程式)

$$y'' - 3y' + 2y = x$$

typing...

□

## § 1.8 演習問題 ~ 定数係数線形方程式

**問 1.25** (定数係数斉次線形方程式) 次の方程式の一般解を求めよ.

- (1)  $y'' - y = 0$  (2)  $y'' - y' - 2y = 0$  (3)  $y'' - y' - 6y = 0$  (4)  $y'' + y = 0$   
 (5)  $y'' + y' - 2y = 0$  (6)  $y'' + y' - 6y = 0$  (7)  $3y'' - y' - 2y = 0$  (8)  $y'' - 2y' + 2y = 0$   
 (9)  $y'' - 2y' - 3y = 0$  (10)  $y'' - 2y' + 5y = 0$  (11)  $2y'' - 2y' + y = 0$   
 (12)  $2y'' - 2y' + 5y = 0$  (13)  $y'' + 2y' - y = 0$  (14)  $y'' + 2y' + y = 0$   
 (15)  $y'' + 2y' + 2y = 0$  (16)  $y'' + 2y' - 3y = 0$  (17)  $y'' + 2y' + 2y = 0$   
 (18)  $y'' + 2y' + 5y = 0$  (19)  $y'' + 2y' + 10y = 0$  (20)  $2y'' + 2y' + y = 0$   
 (21)  $3y'' + 2y' - y = 0$  (22)  $y'' - 3y' + 2y = 0$  (23)  $y'' - 3y' + 3y = 0$   
 (24)  $y'' + 3y' - 4y = 0$  (25)  $2y'' + 3y' + y = 0$  (26)  $y'' - 4y = 0$  (27)  $y'' - 4y' + 3y = 0$   
 (28)  $y'' - 4y' + 4y = 0$  (29)  $y'' + 4y = 0$  (30)  $y'' + 4y' + 4y = 0$  (31)  $y'' + 4y' - 5y = 0$   
 (32)  $y'' + 5y' - 6y = 0$  (33)  $2y'' - 5y' + 3y = 0$  (34)  $2y'' - 5y' + 4y = 0$   
 (35)  $y'' - 6y' + 8y = 0$  (36)  $y'' - 6y' + 9y = 0$  (37)  $y'' - 6y' + 13y = 0$   
 (38)  $y'' + 6y' + 9y = 0$  (39)  $2y'' + 6y' + 5y = 0$  (40)  $y'' - 7y' + 12y = 0$   
 (41)  $y'' - 8y' + 15y = 0$  (42)  $y'' - 8y' + 16y = 0$  (43)  $y'' - 8y' + 32y = 0$   
 (44)  $y'' + 9y = 0$  (45)  $4y'' - 12y' + 9y = 0$  (46)  $y'' - ay' = 0$  ( $a \neq 0$ )  
 (47)  $y'' + a^2y = 0$  ( $a \neq 0$ ) (48)  $y'' - 2\omega y' + \omega^2y = 0$  (49)  $y'' + (\alpha + \beta)y' + \alpha\beta y = 0$   
 (50)  $y''' - 3y'' + 4y = 0$  (51)  $y''' + 3y'' + 4y' + 2y = 0$  (52)  $y'' + 2y' - 3y = 0$   
 (53)  $y''' + 3y'' + 3y' + y = 0$  (54)  $y''' - 3y'' + 2y' = 0$  (55)  $y''' + y'' - 2y' + 12y = 0$   
 (56)  $y''' + y'' + y' - 3y = 0$  (57)  $y''' + 3y'' + 3y' + y = 0$  (58)  $y''' - 3y'' + 2y' = 0$   
 (59)  $y''' - 6y'' + 11y' - 6y = 0$  (60)  $y''' - 2y'' - 5y' + 6y = 0$  (61)  $y''' - 3y'' + 3y' - y = 0$

□

**問 1.26** (定数係数非斉次線形方程式) 次の方程式の一般解を求めよ.

- (1)  $y' - y = \sin x$  (2)  $y' + y = x^2 + x - 1$  (3)  $y' - 2y = xe^{2x}$  (4)  $y'' - y' = e^{2x}$   
 (5)  $y'' - y' - 2y = x$  (6)  $y'' - y' - 2y = x^2 - x$  (7)  $y'' - y' - 2y = \sin x$  (8)  $y'' + y' = e^{-x}$   
 (9)  $y'' + y = cx \sin x$  (10)  $y'' + y' - 2y = be^x$  (11)  $y'' + y' - 2y = be^{-x}$   
 (12)  $y'' + y' - 6y = e^{3x}$  (13)  $y'' + y' - 6y = \cos x$  (14)  $y'' - 2y' + y = e^x$   
 (15)  $y'' - 2y' + 2y = x^2 - 1$  (16)  $y'' - 2y' + 2y = e^x \sin x$  (17)  $y'' - 2y' + 2y = 2e^x \cos x$   
 (18)  $y'' - 2y' - 3y = e^{-x}$  (19)  $y'' - 2y' - 3y = e^{-x} + x$  (20)  $y'' - 2y' + 5y = x$   
 (21)  $y'' - 2y' + 5y = e^x$  (22)  $y'' - 2y' + 5y = \sin x$  (23)  $y'' + 2y' - 3y = x^2$   
 (24)  $y'' + 2y' - 3y = e^{2x}$  (25)  $y'' + 2y' - 3y = \cos x$  (26)  $y'' + 2y' - 3y = e^x \sin x$   
 (27)  $y'' + 2y' - 8y = e^{2x}$  (28)  $y'' - 3y' + 2y = 2$  (29)  $y'' - 3y' + 2y = 2x - 1$   
 (30)  $y'' - 3y' + 2y = e^x$  (31)  $y'' - 3y' + 2y = \cos x$  (32)  $y'' - 3y' + 2y = \cos 2x$   
 (33)  $y'' - 3y' + 2y = 2 \cos x + 3e^x$  (34)  $y'' - 3y' + 2y = xe^x$  (35)  $y'' - 3y' + 3y = e^x$   
 (36)  $y'' + 3y' = e^{3x} + x$  (37)  $2y'' + 3y' + y = ax$  (38)  $y'' - 4y' = x + x^2 + \sin x$   
 (39)  $y'' - 4y' + 3y = x$  (40)  $y'' - 4y' + 3y = x^2$  (41)  $y'' - 4y' + 3y = \cos x$   
 (42)  $y'' - 4y' + 3y = 3e^{2x} + 4e^x + 2e^{3x}$  (43)  $y'' - 4y' + 4y = e^x$   
 (44)  $y'' + 4y' + y = e^x$  (45)  $y'' - 5y' - 4y = x + x^2 + \sin x$  (46)  $y'' - 6y' + 8y = e^{3x}$

- (47)  $y'' - 6y' + 9y = be^{3x}$  (48)  $y'' + 8y' + 17y = 2e^{-3x}$  (49)  $y'' - y = x$   
(50)  $y'' - y = e^x$  (51)  $y'' - y = e^{2x}$  (52)  $y'' - y = \sin x$  (53)  $y'' - y = xe^x$   
(54)  $y'' - y = e^x \sin x$  (55)  $y'' + y = x$  (56)  $y'' + y = \sin x$  (57)  $y'' + y = \cos x$   
(58)  $y'' + y = 2 \cos x$  (59)  $y'' + y = x + x^2 + \sin x$  (60)  $y'' + y = x \sin x$   
(61)  $y'' + y = e^x \sin x$  (62)  $y'' + 2y = x^2$  (63)  $y'' - 4y = 1$  (64)  $y'' - 4y = x$   
(65)  $y'' - 4y = \sin x$  (66)  $y'' - 4y = \cos x$  (67)  $y'' - 4y = e^{2x}$  (68)  $y'' - 4y = e^{-2x}$   
(69)  $y'' - \omega^2 y = a + bx$  (70)  $y'' - \omega^2 y = ce^{\rho x}$  ( $\rho \neq \pm \omega$ ) (71)  $y'' - \omega^2 y = ce^{\omega x}$   
(72)  $y'' - \omega^2 y = a + bx + ce^{\rho x}$  (73)  $y'' + \omega^2 y = bx$   
(74)  $y'' + \omega^2 y = a \sin \Omega x$  ( $\omega \neq \Omega$ ) (75)  $y''' - y = e^{2x}$  (76)  $y''' - 7y' + 6y = e^x$   
(77)  $y''' + y'' + y' + y = e^x$  (78)  $y''' - 3y'' + 4y = e^{2x}$  (79)  $y''' - 5y'' + 6y' = e^{2x}$   
(80)  $y''' - 5y'' + 2y' + 8y = 16x$  □

**問 1.27** (オイラーの方程式) 次の方程式を  $x = e^t$  と変数変換して一般解を求めよ .

- (1)  $x^2 y'' - 2y = 0$  (2)  $x^2 y'' - xy' + y = 0$  (3)  $x^2 y'' - xy' + y = 0$   
(4)  $x^2 y'' - xy' - 3y = 0$  (5)  $x^2 y'' - xy' + 5y = 0$  (6)  $x^2 y'' - xy' + y = (\log x)^2$   
(7)  $xy'' - 2y' = 0$  (8)  $x^2 y'' - 2xy' + 2y = x^3$  (9)  $x^2 y'' - 3xy' + 3y = 0$   
(10)  $x^2 y'' - 3xy' + 3y = x^3 + x^4$  (11)  $x^2 y'' - 3xy' + 3y = x^5 \sin x$  (12)  $x^2 y'' - 3xy' + 4y = 0$   
(13)  $x^2 y'' - 3xy' + 4y = x^3 + x^4$  (14)  $x^2 y'' - 3xy' + 4y = x^2 + x^2 \log x$   
(15)  $x^2 y'' - 4xy' + 6y = 2x$  (16)  $x^2 y'' - 4xy' + 6y = 0$  (17)  $x^2 y y'' + x^2 y'^2 - x y y' = 0$   
(18)  $x^2 y'' - (2m - 1)xy' + m^2 y = 0$  (19)  $x^2 y'' - (\alpha + \beta - 1)xy' + \alpha \beta y = 0$   
(20)  $y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$  (21)  $y'' - \frac{\alpha}{x} y' + \frac{\alpha}{x^2} y = 0$   
(22)  $y'' - \frac{y'}{x} + \frac{y}{x^2} = bx^\alpha$  ( $\mathbb{R} \ni \alpha \neq 1$ ) □

## § 1.9 1 階線形方程式

**例 1.28** (1 階線形方程式)

$$y' - y = x$$

typing...

□

**例 1.29** (1 階線形方程式)

$$y' - ay = q(x)$$

typing...

□

## § 1.10 ベルヌーイの方程式

例 1.30 (ベルヌーイの方程式)

$$y' + y = xy^3$$

typing...

□

## § 1.11 演習問題 ~ 1階線形, ベルヌーイ型

**問 1.31** (同次型) 次の方程式の一般解を求めよ.

(1)  $y' = y + e^{2x}$  (2)  $xy' = 2y + x(x+2)$  (3)  $y' + 2xy = x$  (4)  $xy' = x + y$

(5)  $y' = y + xy^2$  (6)  $y' = 2y + e^{2x}y^3$  (7)  $y' + y = x$  (8)  $xy' + y = x$

(9)  $xy' + y = x^2$  (10)  $y' = y + \sin x$  (11)  $y' = xy + x$  (12)  $y' \cos x + y \sin x = 1$

(13)  $xy' + y = \cos x$  (14)  $y' + \cos^2 y \tan y = x \cos^2 y$

□

## § 1.12 演習問題 ~ 巾級数展開

**問 1.32** (巾級数展開) 次の方程式の一般解を  $x = 0$  に巾級数展開で求めよ .

- (1)  $y'' - 3y' + 2y = 0$  (2)  $y'' - xy' - y = 0$  (3)  $(x^2 - 1)y'' - 2y = 0$   
(4)  $(1 - x^2)y'' - 2xy' + 2y = 0$  (5)  $y' = 1 + x + 2y$  (6)  $xy' = x^2 + y$   
(7)  $y'' - 2xy' - 2y = 0$  (8)  $(x^2 - 1)y'' + 2xy' - 2y = 0$  (9)  $(1 + x^2)y'' - xy' - 8y = 0$   
(10)  $y'' + x^2y' + xy = 0$  (11)  $y' - 2y = 1 + x$  (12)  $y' - y/x = x$  (13)  $y'' - xy = 0$   
(14)  $(1 - x^2)y' - xy' + y = 0$  (15)  $(1 - x)y' = -1 + y$  (16)  $(1 + x^2)y'' + xy' - y = 0$   
(17)  $y' - y + x = 0$  (18)  $y' - 2xy = x$  (19)  $y' + y = x^2$  (20)  $(x + 1)y' + y = 2x + 3x^2$   
(21)  $y'' + y = 0$  (22)  $y'' - y = 2e^x$  (23)  $(1 - x)y'' + xy' - y = 0$   
(24)  $(1 + x^2)y'' - 2xy' + 2y = 0$  □

**問 1.33** (確定特異点をもつ方程式の巾級数展開) 次の方程式の一般解を  $x = 0$  に巾級数展開で求めよ .

- (1)  $4xy'' + 2y' + y = 0$  (2)  $x^2y'' + (x^2 - 3x)y' + (4 - 2x)y = 0$   
(3)  $x^2(1 - x)y'' - x(2 + x)y' + 2y = 0$  (4)  $xy'' + y' + xy = 0$  (5)  $2xy'' + (1 - 2x)y' - y = 0$   
(6)  $xy'' + y' - y = 0$  (7)  $x^2y'' + xy' + (x^2 - 1/4)y = 0$  (8)  $xy'' - xy' - y = 0$  □